

ON COLIMITS AND ELEMENTARY EMBEDDINGS

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ABSTRACT. We give a sharper version of a theorem of Rosický, Trnková and Adámek [12], and a new proof of a theorem of Rosický [13], both about colimits in categories of structures. Unlike the original proofs, which use category-theoretic methods, we use set-theoretic arguments involving elementary embeddings given by large cardinals such as α -strongly compact and $C^{(n)}$ -extendible cardinals.

1. INTRODUCTION

Many problems in category theory, homological algebra, and homotopy theory have been shown to be set-theoretical, involving the existence of large cardinals. For example, the problem of the existence of rigid classes in categories such as graphs, or metric spaces or compact Hausdorff spaces with continuous maps, which was studied by the Prague school in the 1960's turned out to be equivalent to the large cardinal principle now known as Vopěňka's Principle (VP) (see [7]). Another early example is John Isbell's 1960 result that \mathbf{Set}^{op} is bounded if and only if there is no proper class of measurable cardinals. A summary of these and similar results can be found in the monograph [10]. In the 1980's, many statements in category theory previously known to hold under the assumption of VP were shown to be actually equivalent to it. The following is a small sample of such statements (see [1] for an excellent survey, with complete proofs, of these and many other equivalence results):

- (1) The category **Ord** of ordinals cannot be fully embedded into the category **Gra** of graphs.
- (2) A category is locally presentable if and only if it is complete and bounded.
- (3) A category is accessible if and only if it is bounded and has λ -directed colimits for some regular cardinal λ .
- (4) Every subfunctor of an accessible functor is accessible.
- (5) For every full embedding $F : \mathcal{A} \rightarrow \mathcal{K}$, where \mathcal{K} is an accessible category, there is a regular cardinal λ such that F preserves λ -directed colimits.

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Even though each one of these statements is equivalent to VP, their known proofs from VP use category-theoretic, rather than set-theoretic, methods and arguments.

Recently, new equivalent formulations of VP in terms of elementary embeddings have been used in [3] to improve on previous results in category theory and homotopy theory. For the reader unfamiliar with them, an *elementary embedding* $j : V \rightarrow M$ is a function that preserves *all* formulas: for every formula φ with parameters a_1, \dots, a_n in V , $V \models \varphi(a_1, \dots, a_n)$ if and only if $M \models \varphi(j(a_1), \dots, j(a_n))$, where \models denotes the model satisfaction relation. Many of the strongest large cardinal axioms are most naturally expressed in these terms, with the pertinent cardinal being the *critical point* $\text{crit}(j)$ of the embedding, that is, the least cardinal κ for which $j(\kappa) \neq \kappa$.

In [3], these elementary embedding formulations are used with a set-theoretic analysis to improve previous results by showing that much weaker large cardinal assumptions suffice for them. Specifically, the necessary large cardinal hypothesis in each case depends on the complexity of the formulas defining the categories involved, in the sense of the Lévy hierarchy. For example, the statement that, for a class \mathcal{S} of morphisms in an accessible category \mathcal{C} , the orthogonal class of objects \mathcal{S}^\perp is a small-orthogonality class is provable in ZFC if \mathcal{S} is Σ_1 , it follows from the existence of a proper class of supercompact cardinals if \mathcal{S} is Σ_2 , and from the existence of a proper class of $C^{(n)}$ -extendible cardinals if \mathcal{S} is Σ_{n+2} for $n \geq 1$. These cardinals form a hierarchy, and VP is equivalent to the existence of $C^{(n)}$ -extendible cardinals for all n (see also [2]). As a consequence, the existence of cohomological localizations of simplicial sets, a long-standing open problem in algebraic topology solved in [5] assuming VP, follows just from the existence of sufficiently large supercompact cardinals.

In this paper we continue this programme of giving sharper versions of results in category theory, in this case results about colimits in accessible categories, and more generally in categories of structures in infinitary languages. What is different in our work, however, is that we use the elementarity of the embeddings in a strong way. In previous work in this context, once one has obtained an elementary embedding it has generally been used as little more than a convenient homomorphism. By contrast, we shall make great use of elementarity to make our proofs work, as we move between various set-ups and their images under the embedding. It is our belief that similar “strong uses” of elementarity will lead to many improvements and new results in the area.

The first main result we consider (Theorem 10 below) is due to J. Rosický [13, Theorem 1 and Remark 1 (2)], which in turn generalizes an earlier result of Richter [11]. The proof given by Rosický uses atomic diagrams, explicit ultraproducts, and something called “purity”; our proof using elementary embeddings and ideas from the original paper of Richter seems much cleaner.

The second one we consider is item (5) in the list given above of statements equivalent to VP, a result of Rosický, Trnková and Adámek from [12] (see also [1, Theorem 6.9]). In Theorem 18 below we prove a result that is simultaneously more general and more refined, using appropriate fragments of VP. Specifically, we use $C^{(n)}$ -extendible cardinals, with n determined by the complexity of the definitions of the categories involved. Further, we are able to show (Theorem 20) that the $C^{(n)}$ -extendible cardinals are necessary in an almost, but unfortunately not exactly, level-by-level equivalence.

We would like to remark that while the proofs of the two theorems in the original papers are quite different from each other in their methods, our new proofs turn out to be quite similar. We think this shows that our set-theoretic arguments using elementary embeddings are more generally applicable and more natural in this context.

2. PRELIMINARIES

As is standard, we denote by V the universe of all sets. For all of our elementary embeddings $j : V \rightarrow M$, M is a class of V , so we can in effect treat j as a functor from V to V (but note that it is only *elementary* as a map with codomain M). The notation $j^{\mathcal{D}}$ (j point-wise on \mathcal{D}) denotes the diagram consisting of the objects $j(d)$ for d an object of \mathcal{D} , and the morphisms $j(f)$ for f a morphism of \mathcal{D} . Note in particular that this will generally be different from $j(\mathcal{D})$, that is, j applied to the diagram \mathcal{D} as a whole: $j(\mathcal{D})$ will generally contain many more objects and morphisms than just those that are images of objects or morphisms from \mathcal{D} , that is, those in $j^{\mathcal{D}}$.

Throughout the paper an important role will be played by the category $\mathbf{Str} \Sigma$ of all Σ -structures for a signature Σ . Here a *signature* is a set of function and relation symbols with associated arities; formally a Σ -structure is something of the form $\langle A, \mathcal{I} \rangle$ where A is a set (the underlying set of the structure) and \mathcal{I} is a function with domain Σ (the interpretation function) such that for each n -ary relation symbol R , $\mathcal{I}(R) \subseteq A^n$ (considered to be the set of tuple where R holds), and for each n -ary function symbol f , $\mathcal{I}(f)$ is a function from A^n to A . We abuse notation identifying A with $\langle A, \mathcal{I} \rangle$, and we shall often write R^A or f^A for $\mathcal{I}(R)$ or $\mathcal{I}(f)$ respectively.

Associated to any signature, we have a language of formulas with functions and relations from that signature; see for example [1, Sections 5.2 and 5.24]. Note that we do not constrain ourselves to signatures in which all of the arities are finite; in line with this, we also consider infinitary languages. In this setting, $\mathcal{L}_\lambda(\Sigma)$ denotes the language in which in the definition of formulas we allow conjunctions, disjunctions, and universal and existential quantifications of size less than λ . Thus, $\mathcal{L}_\omega(\Sigma)$ is the usual language over Σ with only finitary conjunctions, disjunctions and quantifications; note however that if Σ contains any symbols of infinite arity, then $\mathcal{L}_\omega(\Sigma)$ cannot have any fully quantified sentences involving those symbols. The notion of a structure A satisfying a formula φ in $\mathcal{L}_\lambda(\Sigma)$, denoted $A \models \varphi$, is defined in line with the expected meaning — see for example [1, Sections 5.3 and 5.26].

Whilst Σ may be infinite and infinitary, its basic logical role means that we generally do not want it to be affected by the elementary embeddings we employ, which are only elementary for the language of set theory, $\mathcal{L}_\omega(\{\in\})$. Indeed in Section 4 we make assumptions to this effect. However, in Section 3, we can in fact handle a Σ large enough to be changed by the embedding, so long as the arity of each individual symbol is small enough to be unaffected. To this end, let us call a signature Σ λ -ary if every symbol in Σ has arity strictly less than λ .

Definition 1. Suppose j is an elementary embedding from V to M with critical point κ , Σ is a κ -ary signature, and C is a $j(\Sigma)$ -structure. Then C_Σ is the Σ -structure obtained by reducing C to C' over signature $j^{\mathcal{D}}\Sigma$, and then considering

the interpretation of $j(R)$ in C' to be the interpretation of R in C_Σ , for any symbol R in Σ .

Here “reducing” is in the model-theoretic sense of simply “forgetting” those function and operation symbols in $j(\Sigma)$ but not $j\text{“}\Sigma$ (and leaving the underlying set unchanged). Thus, the structure $\langle C, R^C \mid R \in j(\Sigma) \rangle$ reduces to $\langle C, R^C \mid R \in j\text{“}\Sigma \rangle$, and then it is simply a matter of relabelling the indices to consider this to be $\langle C, j(S)^C \mid S \in \Sigma \rangle = C_\Sigma$.

We give some very basic lemmas about this notion. For all of them, take j , κ , and Σ as in Definition 1.

Lemma 2. *For any $\lambda < \kappa$ and any theory T for the language $\mathcal{L}_\lambda(\Sigma)$, if C is a model of $j(T)$, then C_Σ is a model of T .*

Proof. This is immediate from the recursive definition of \models , as presented for example in [1, Section 5.26]. \square

Note however that with Lemma 2 we are *not* claiming that $M \models \text{“}C \text{ is a model of } j(T)\text{”}$ implies that $V \models \text{“}C_\Sigma \text{ is a model of } T\text{”}$. For this further step we shall need M to contain all of the tuples of length less than λ of elements of C , so that the statement “ C_Σ is a model of T ” is absolute from M to V .

Lemma 3. *For any $\lambda < \kappa$ and any theory T for the language $\mathcal{L}_\lambda(\Sigma)$, if A is a model of T , then $j(A)_\Sigma$ is a model of T .*

Proof. This is immediate from Lemma 2 by elementarity. \square

Lemma 4. *Let \cdot_Σ denote the function that takes $j(\Sigma)$ -structures C to C_Σ and takes $j(\Sigma)$ -homomorphisms to Σ -homomorphisms by leaving them unchanged on the underlying set of the domain. Then \cdot_Σ is a functor from the category of $j(\Sigma)$ -structures to the category of Σ -structures.* \square

We shall studiously include Σ subscripts in our notation for objects, but omit them from homomorphisms, since they have no effect on them as functions.

We denote by **Str** Σ the category of all Σ -structures with homomorphism as the morphisms, and by **Set** the category of all sets with arbitrary functions as the morphisms. In [13], Rosický allows for a change of language. We note that, much as Theorem 10 seems to say that the theory is irrelevant for κ -directed colimits, so too is the language, in the following sense. For any κ -ary signature Σ and any κ -directed diagram in **Str** Σ , the colimit of the diagram exists, and is the direct limit of the structures. In particular, the underlying set of the colimit is simply the colimit in **Set** of the diagram of the underlying sets, and the interpretations in the colimit of the relation and function symbols of Σ are then uniquely determined (in the terminology of [9], the forgetful functor from **Str** Σ to **Set**, which takes each Σ -structure to its underlying set, *creates* κ -directed colimits). Here κ -directedness is required so that every term $f(\mathbf{a})$ in the colimit (for $f \in \Sigma$) appears in one of the structures of the diagram — one that contains every component of \mathbf{a} . In fact **Set** is simply the special case in which we have reduced to empty signature: we likewise have that for any extension κ -ary signature $\Sigma' \supseteq \Sigma$, the reduction functor from **Str** Σ' to **Str** Σ preserves λ -directed colimits. In particular, to obtain [13, Theorem 1], it suffices to consider a single language, which is the approach we take here.

We denote by $(j \upharpoonright \cdot)$ the natural transformation that associates to A the function $j \upharpoonright A : A \rightarrow j(A)_\Sigma$. Of course, we should check that it is indeed a natural transformation.

Lemma 5. *For any elementary embedding $j : V \rightarrow M$ with critical point κ and any λ -ary signature Σ for some $\lambda < \kappa$, $(j \upharpoonright \cdot)$ is a natural transformation from the identity functor to the functor $\cdot_\Sigma \circ j$. That is, for any Σ -structure homomorphism $f : A \rightarrow B$, the following diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{j \upharpoonright A} & j(A)_\Sigma \\ f \downarrow & & \downarrow j(f) \\ B & \xrightarrow{j \upharpoonright B} & j(B)_\Sigma \end{array}$$

Moreover, if M is closed under tuples of length less than λ , then each morphism $j \upharpoonright A$ is an elementary embedding from A to $j(A)_\Sigma$.

Proof. First note that for any A , $j \upharpoonright A$ is a Σ -structure homomorphism to $j(A)_\Sigma$: if $A \models R(\langle a_i \mid i \in \alpha \rangle)$, then by elementarity $j(A) \models j(R)(\langle j(a_i) \mid i \in \alpha \rangle)$ (in M , but this is absolute, so also in V), so $j(A)_\Sigma \models R(\langle j(a_i) \mid i \in \alpha \rangle)$. The corresponding statement holds for equations involving function symbols, showing that $j \upharpoonright A$ is a homomorphism. Further, if M is closed under tuples of length less than λ , then for any first order formula φ in the language Σ , $j(A) \models \varphi(\langle j(a_i) \mid i \in \alpha \rangle)$ in M if and only if $j(A) \models \varphi(\langle j(a_i) \mid i \in \alpha \rangle)$ in V , since this satisfaction statement is Δ_1 in the parameter $j(A)^{<\lambda}$, and hence absolute between such M and V . Indeed, equivalent Σ_1 and Π_1 definitions may be extracted from the usual recursive definition of \models for set-sized models, as given for example in [1, Section 5.26]. This shows that, entirely in V , $j \upharpoonright A$ is an elementary embedding from A to $j(A)$.

Now for any $a \in A$, $M \models j(f)(j(a)) = j(f(a))$, by elementarity. But this statement is also absolute, and so also true in V . \square

Penultimately for this section, we enunciate a simple observation that will be useful.

Lemma 6. *Suppose \mathcal{C}_0 is a full subcategory of \mathcal{C}_1 and \mathcal{D} is a diagram in \mathcal{C}_0 . If a colimit C of \mathcal{D} in \mathcal{C}_1 exists and lies in \mathcal{C}_0 , then C is also a colimit of \mathcal{D} in \mathcal{C}_0 , with the same colimit cocone.*

Finally, the following definition is useful in the discussion both of infinitary languages and of large cardinals.

Definition 7. *For any set X and any cardinal κ , $\mathcal{P}_\kappa(X)$ denotes the set of all subsets of X of cardinality less than κ .*

3. COLIMITS OF STRUCTURES AND MODELS

Theorem 10 below is due to J. Rosický [13, Theorem 1 and Remark 1 (2)]. The proof given by Rosický uses atomic diagrams, explicit ultraproducts, and something called “purity”; we avoid all that, using elementary embeddings and ideas from the original paper of Richter [11] that Rosický’s theorem extends. We stick reasonably closely to Rosický’s notation in our proof of the theorem, but note that we use \mathcal{D} for the diagram rather than D so that non-caligraphic uppercase Roman letters near

the start of the alphabet are always objects in one of our two main categories (T -models and Σ -structures). Compositions of homomorphisms with cocones have the obvious meaning (as do compositions on the other side of natural transformations with cocones).

The large cardinal axiom required for Theorem 10 is the following.

Definition 8. *If $\alpha \leq \kappa$ are uncountable cardinals, then we say that κ is α -strongly compact if for every set X , every κ -complete filter on X can be extended to an α -complete ultrafilter on X . The cardinal κ is strongly compact if it is κ -strongly compact.*

Note that if $\kappa \leq \lambda$ and κ is α -strongly compact, then λ is also α -strongly compact. In particular, α -strongly compact cardinals can be singular; but even further, the least α -strongly compact cardinal can be singular, as was shown in [4] for $\alpha = \omega_1$ under the assumption of the consistency of the existence of a supercompact cardinal. We shall make use of the following characterization of α -strongly compact cardinals in terms of elementary embeddings.

Theorem 9 ([4, Theorem 4.7]). *The following are equivalent for any uncountable cardinals $\alpha < \kappa$:*

- (1) κ is α -strongly compact.
- (2) *For every γ greater than or equal to κ there exists an elementary embedding $j : V \rightarrow M$ definable in V , with M transitive, ${}^\alpha M \subset M$, $\text{crit}(j) \geq \alpha$, and such that there exists $Z \in M$ with $j''\gamma = \{j(\beta) : \beta < \gamma\} \subseteq Z$ and $M \models |Z| < j(\kappa)$.*
- (3) *For every cardinal $\gamma > \kappa$, there exists an α -complete fine ultrafilter on $\mathcal{P}_\kappa(\gamma)$.*

Here a *fine* ultrafilter \mathcal{U} on $\mathcal{P}_\kappa(\gamma)$ is one such that for every $\alpha \in \gamma$,

$$\{X \in \mathcal{P}_\kappa(\gamma) \mid \alpha \in X\} \in \mathcal{U}.$$

In the case of a regular κ , such an ultrafilter can be obtained using α -strong compactness by extending the κ -complete filter generated by such sets. The embedding j in (2) is obtained by taking the ultrapower of V by an α -complete fine ultrafilter \mathcal{U} on $\mathcal{P}_\kappa(\gamma)$ as in (3).

Theorem 10. *Let λ be an infinite cardinal, T a theory for the language $\mathcal{L}_\lambda(\Sigma)$ over λ -ary signature Σ , and suppose there exists a cardinal κ that is α -strongly compact, where $\alpha = \max\{\lambda, \omega_1\}$. Suppose \mathcal{D} is a κ -directed diagram of models for T , and suppose it has a colimit in the category $\mathbf{Mod} T$ of models of T . The diagram \mathcal{D} may also be considered to be a diagram in $\mathbf{Str} \Sigma$; let A be its colimit in this category. Then A is a model of T . Hence, A is the colimit of \mathcal{D} in $\mathbf{Mod} T$.*

Note that the assumption that a $\mathbf{Mod} T$ colimit does exist is important — see Example 11 below.

Proof. Since $\mathbf{Mod} T$ is a subcategory of $\mathbf{Str} \Sigma$, we will freely consider \mathcal{D} to be a diagram in either as the context requires. Let $\delta^A : \mathcal{D} \rightarrow A$ be the colimit cocone to A as colimit of \mathcal{D} in $\mathbf{Str} \Sigma$. Let B denote the colimit of \mathcal{D} in $\mathbf{Mod} T$, and let $\delta^B : \mathcal{D} \rightarrow B$ denote the colimit cocone. Since δ^B is in particular a Σ -structure cocone, there is a unique Σ -structure homomorphism h from A to B such that $h \circ \delta^A = \delta^B$.

The proof starts by chasing around the following diagram, in which bold font denotes diagrams, and double-stemmed arrows (\Rightarrow) are used for cocones, natural transformations generally, and the inclusion $(j^{\text{“}}\mathcal{D})_{\Sigma}$ to $j(\mathcal{D})_{\Sigma}$.

$$\begin{array}{ccccc}
 (*) & A & \xrightarrow{j \upharpoonright A} & j(A)_{\Sigma} & \\
 & \delta^A \swarrow & g_A \searrow & \nearrow j(\delta^A)_{\bar{A}} & \\
 & B & \xrightarrow{j \upharpoonright B} & j(B)_{\Sigma} & \\
 & \delta^B \swarrow & g_B \searrow & \nearrow j(\delta^B)_{\bar{A}} & \\
 \mathcal{D} & \xrightarrow{(j \upharpoonright \cdot)} & (j^{\text{“}}\mathcal{D})_{\Sigma} & \xrightarrow{\text{inclusion}} & j(\mathcal{D})_{\Sigma}
 \end{array}$$

$\begin{array}{c} \delta^A \\ \delta^B \end{array}$
 $\begin{array}{c} h \\ \zeta \end{array}$
 $\begin{array}{c} j(h) \\ j(\delta^A) \\ j(\delta^B) \end{array}$

First note that h is epi for homomorphisms to T -models. That is, if $f, g : B \rightarrow C$ are homomorphisms with codomain C a T -model such that $f \circ h = g \circ h$, then $f = g$. For, considering cocones, we have $f \circ \delta^B = f \circ h \circ \delta^A = g \circ h \circ \delta^A = g \circ \delta^B$, from which the uniqueness-of-factorisation property of B as a colimit gives $f = g$.

Let γ be the number of objects in \mathcal{D} . Let $j : V \rightarrow M$ be an ultrapower embedding for an α -complete ultrafilter over $\mathcal{P}_{\kappa}(\gamma)$, as in (2) of Theorem 9, so that the critical point of j is at least α , j is definable in V , and there exists $Z \in M$ such that $j^{\text{“}}\gamma = \{j(\beta) : \beta < \gamma\} \subseteq Z$ and $M \models |Z| < j(\kappa)$. Moreover, M is closed under α -tuples, so by the same argument as in the proof of Lemma 5, being a model of T is absolute between M and V , and for every Σ -structure C , $j \upharpoonright C$ is elementary from C to $j(C)_{\Sigma}$.

Since \mathcal{D} is a κ -directed diagram of models of T ,

$$M \models j(\mathcal{D}) \text{ is a } j(\kappa)\text{-directed diagram of models of } j(T).$$

The objects in $j(\mathcal{D})$ are $j(\Sigma)$ -structures, which may be thought of as Σ -structures using \cdot_{Σ} . Since $Z \in M$, $j^{\text{“}}\gamma \subseteq Z$, and $M \models |Z| < j(\kappa)$, we have that for any subset X of M of size at most γ , there is a $Y \in M$ such that $Y \supseteq X$ and $M \models |Y| < j(\kappa)$ (see for example Kanamori [7, Theorem 22.4]). In particular, since $j^{\text{“}}\mathcal{D}$ has cardinality γ (the same as \mathcal{D}), there is a $Y \in M$ with $j^{\text{“}}\mathcal{D} \subseteq Y$ and $M \models |Y| < j(\kappa)$. Intersecting such a Y with $j(\mathcal{D})$ and applying $j(\kappa)$ -directedness in M , we conclude that there is an object \bar{A} in $j(\mathcal{D})$ such that there are $j(\Sigma)$ -homomorphisms to \bar{A} from every object in $j^{\text{“}}\mathcal{D}$, yielding a cocone ζ from the subdiagram $(j^{\text{“}}\mathcal{D})_{\Sigma}$ of $j(\mathcal{D})_{\Sigma}$ to \bar{A}_{Σ} (note: ζ as a whole is not necessarily in M , although each component homomorphism of it is). Using Lemma 5, these maps compose with the natural transformation $(j \upharpoonright \cdot)$ to give a cocone from \mathcal{D} to \bar{A}_{Σ} in V . Using the colimit definition of A , let $g_A : A \rightarrow \bar{A}_{\Sigma}$ be the unique Σ -structure homomorphism such that $g_A \circ \delta^A = \zeta \circ (j \upharpoonright \cdot) : \mathcal{D} \rightarrow \bar{A}_{\Sigma}$. Moreover, since \bar{A} is in $j(\mathcal{D})$, \bar{A} is a model of $j(T)$, so \bar{A}_{Σ} is a model of T , and hence there is likewise a unique homomorphism $g_B : B \rightarrow \bar{A}_{\Sigma}$ of models of T such that $g_B \circ \delta^B = \zeta \circ (j \upharpoonright \cdot)$. By the uniqueness of factorisation through δ^A , we have $g_B \circ h = g_A$. Also, there is a $j(\Sigma)$ -structure map from \bar{A} to $j(A)$, namely the \bar{A} component $j(\delta^A)_{\bar{A}}$ of the colimit cocone $j(\delta^A)$ from $j(\mathcal{D})$ to $j(A)$ in the category of Σ -structures of M (of course, this is the colimit cocone in M by elementarity). Likewise, we have the \bar{A} component $j(\delta^B)_{\bar{A}} : \bar{A} \rightarrow$

$j(B)$ of the colimit cocone $j(\delta^B)$ from $j(\mathcal{D})$ to $j(B)$ in the category of T models of M . Applying \cdot_Σ , we get Σ -structure maps from \bar{A}_Σ to $j(A)_\Sigma$ and $j(B)_\Sigma$.

There are two maps from A to $j(A)_\Sigma$ that arise naturally: $j \upharpoonright A$, and the map that exists because A is the colimit of \mathcal{D} , induced by the cocone $j(\delta^A) \circ (j \upharpoonright \cdot)$. By uniqueness, the latter map equals $j(\delta^A)_{\bar{A}} \circ g_A$. By considering the concrete construction of the colimits A and $j(A)$, we see that these two maps are in fact the same: an element of A given as $[a]$, the equivalence class of an element a of some $D_i \in \mathcal{D}$, must be mapped in each case to the element $[j(a)] \in j(A)$.

Similarly, consider $j \upharpoonright B$, and the colimit map from B to $j(B)_\Sigma$ induced by the cocone $j(\delta^B) \circ (j \upharpoonright \cdot)$, which equals $j(\delta^B)_{\bar{A}} \circ g_B$. In this case we cannot appeal to a concrete construction of B to show that they are the same. However, their respective compositions with h are both equal to $j(h) \circ j \upharpoonright A$: by Lemma 5 (that is, by elementarity of j) in the case of $j \upharpoonright B \circ h$, and by uniqueness of the map from A to $j(B)_\Sigma$ factorising the relevant cocone in the case of $j(\delta^B)_{\bar{A}} \circ g_B \circ h$. Since h is epi for homomorphisms to T -models as noted above, and $j(B)_\Sigma$ is a model of T , it follows that $j \upharpoonright B = j(\delta^B)_{\bar{A}} \circ g_B$.

We may therefore conclude that diagram (*) above commutes. Applying j repeatedly, we now get a commutative diagram

$$\begin{array}{ccccccc}
 A & \xrightarrow{j \upharpoonright A} & j(A)_\Sigma & \xrightarrow{j(j \upharpoonright A)} & j^2(A)_\Sigma & \xrightarrow{j^2(j \upharpoonright A)} & \dots \\
 & \searrow h & \nearrow & \searrow j(h) & \nearrow & \searrow j^2(h) & \nearrow \\
 & B & \xrightarrow{j \upharpoonright B} & j(B)_\Sigma & \xrightarrow{j(j \upharpoonright B)} & j^2(B)_\Sigma & \longrightarrow \dots
 \end{array}$$

with the horizontal mappings all elementary embeddings. The direct limits of the top and bottom horizontal chains therefore give structures into which A and B respectively embed elementarily. But since the chains interleave, the direct limits must be the same. Therefore, the (complete) theory of A is the same as that of B , and in particular, A is a model of T . \square

The following example shows that the assumption that there is a colimit in $\mathbf{Mod} T$ was necessary.

Example 11. Let $\Sigma = \{<\}$, where as usual $<$ is a binary relation, and let T be the theory of linear orders with a maximum element. Let \mathcal{D} be the diagram whose objects are all ordinals less than κ , and whose morphisms are the usual inclusions. Then \mathcal{D} is κ -directed, and its colimit in $\mathbf{Str} \Sigma$ is κ , but \mathcal{D} has no colimit in $\mathbf{Mod} T$.

One might wonder if $\kappa + 1$ could be a colimit for \mathcal{D} in $\mathbf{Mod} T$ in Example 11, but since we are not naming the maximum element with a constant, the uniqueness property of colimits rules this out: for example, there are two order preserving functions from $\kappa + 1$ to $\kappa + 2$ preserving the inclusions of the ordinals less than κ .

4. COLIMITS OF MORE GENERAL CATEGORIES

An important general notion in category theory encompassing many categories of interest is that of an *accessible category*. An accessible category is a category that, for some regular cardinal λ , has λ -directed colimits and a *set* of nice (specifically, λ -presentable) objects which generate the category by λ -directed colimits — see [1, Chapter 2] for precise details.

A large cardinal axiom that has found great applicability in the study of accessible categories (see for example [1, Chapter 6]) is the following.

Definition 12. *Vopěnka's Principle (VP) is the statement that for every signature Σ and every proper class \mathcal{C} of Σ -structures, there are two members A and B of \mathcal{C} such that there exists a (non-identity) elementary embedding $j : A \rightarrow B$.*

Note that quantifying over proper classes is not permitted in standard ZFC set theory, so we treat VP as an axiom schema, giving an axiom for each formula that defines a proper class.

In [12] (see also [1, Theorem 6.9]), Rosický, Trnková and Adámek prove the following colimit preservation theorem for accessible categories.

Theorem 13. *Assuming Vopěnka's Principle, for each full embedding $F : \mathcal{A} \rightarrow \mathcal{K}$, where \mathcal{K} is an accessible category, there exists a regular cardinal λ such that F preserves λ -directed colimits.*

The conclusion of Theorem 13 is in fact equivalent to VP as shown by Example 6.12 of [1].

In Theorem 18 we prove a result that is simultaneously more general and more refined, using appropriate fragments of VP. Specifically, we use $C^{(n)}$ -extendible cardinals.

We recall the *Lévy hierarchy* of formulas. A formula is said to be Σ_0 , Π_0 , or Δ_0 if it involves no unbounded quantifiers. For $n > 0$, a formula (and the notion it expresses) is said to be Σ_n if it is of the form $\exists x(\varphi(x))$ for some $\varphi \in \Pi_{n-1}$, and Π_n if it is of the form $\forall x(\varphi(x))$ for some $\varphi \in \Sigma_{n-1}$. A notion is said to be Δ_n if it can be expressed equivalently by a formula that is Σ_n or a formula that is Π_n . For example, if ZFC implies that there will be a unique x with some property φ , then $\exists x(\varphi(x) \wedge \psi(x))$ and $\forall x(\varphi(x) \implies \psi(x))$ will be equivalent.

Following [2], we denote by $C^{(n)}$ the closed and unbounded proper class of cardinals λ such that V_λ is a Σ_n -elementary substructure of V , that is, a Σ_n statement is true in V_λ if and only if it is true in V . A cardinal κ is called $C^{(n)}$ -extendible if for every $\lambda > \kappa$, there is an elementary embedding $j : V_\lambda \rightarrow V_\mu$ for some $\mu > \lambda$, with critical point κ and with $j(\kappa)$ a cardinal in $C^{(n)}$ greater than λ . A natural strengthening of this is also considered: following [2], we say that a cardinal is $C^{(n)+}$ -extendible if for every $\lambda > \kappa$ in $C^{(n)}$ there is an elementary embedding $j : V_\lambda \rightarrow V_\mu$ for some $\mu > \lambda$ in $C^{(n)}$, with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$ in $C^{(n)}$ (this was actually the definition of $C^{(n)}$ -extendible cardinal used in [3]). Note that every $C^{(n)+}$ -extendible cardinal κ is $C^{(n)}$ -extendible, as for every ordinal $\xi > \kappa$ there is a cardinal λ greater than ξ in $C^{(n)}$, and if $j : V_\lambda \rightarrow V_\mu$ witnesses the λ - $C^{(n)+}$ -extendibility of κ , then $j \upharpoonright V_\xi$ witnesses the ξ - $C^{(n)}$ -extendibility of κ . The two definitions are in fact even more closely related.

Proposition 14. *For every cardinal α , the statement “there is a $C^{(n)}$ -extendible cardinal greater than α ” is equivalent to “there is a $C^{(n)+}$ -extendible cardinal greater than α ”*

Proof. The proof is evident from a careful reading of [2]. Theorem 4.11 of [2] shows that if κ is $C^{(n)}$ -extendible then Vopěnka's Principle holds for Σ_{n+2} -definable proper classes with parameters in V_κ . The proof of Theorem 4.12 of [2], modified as described in the remarks that follow it in that paper, then shows that for every $\xi < \kappa$, there is a $C^{(n)+}$ -extendible cardinal greater than ξ . Thus, “ κ is $C^{(n)}$ -extendible”

implies that “for all $\xi < \kappa$, there is a $C^{(n)+}$ -extendible cardinal greater than ξ ”, which is clearly equivalent to the substantive direction of the Proposition. \square

We can in fact strengthen this further.

Proposition 15. *If a cardinal is $C^{(n)}$ -extendible, then it is either $C^{(n)+}$ -extendible, or it is a limit of $C^{(n)+}$ -extendible cardinals.*

Proof. Suppose κ is $C^{(n)}$ -extendible but not the limit of $C^{(n)+}$ -extendible cardinals. Then there is some $\xi < \kappa$ such that for all ζ strictly between ξ and κ , ζ is not $C^{(n)+}$ -extendible. By Proposition 14, there is a $C^{(n)+}$ -extendible cardinal λ greater than ξ , and hence greater than or equal to κ . If $\lambda = \kappa$ we are done, so suppose $\lambda > \kappa$. We may assume without loss of generality that λ is the *least* $C^{(n)+}$ -extendible cardinal strictly greater than κ . Since λ is $C^{(n)}$ -extendible it is in $C^{(n+2)}$ ([2, Proposition 3.4]), and so since “ κ is $C^{(n)}$ -extendible” and “ κ is $C^{(n)+}$ -extendible” are Π_{n+2} statements (again, see [2]), we have that

$$V_\lambda \models \kappa \text{ is } C^{(n)}\text{-extendible} \wedge \forall \zeta \left((\zeta > \xi \wedge \zeta \neq \kappa) \implies \zeta \text{ is not } C^{(n)+}\text{-extendible} \right).$$

But now λ is also inaccessible, so full ZFC holds in V_λ , and in particular Proposition 14. Thus we may deduce that $V_\lambda \models \kappa$ is $C^{(n)+}$ -extendible, whence by Σ_{n+2} -correctness of V_λ again we have that κ is $C^{(n)+}$ -extendible. \square

This of course raises a natural question.

Open Problem. *Is it consistent to have a $C^{(n)}$ -extendible cardinal that is not $C^{(n)+}$ -extendible?*

It is shown in [2] that VP is equivalent to the existence of a proper class of $C^{(n)}$ -extendible cardinals for every n . Moreover this is a precise stratification, with the existence of a $C^{(n)}$ -extendible cardinal κ corresponding to VP for Σ_{n+2} -definable classes with parameters in V_κ . We now show that this same stratification is applicable to Theorem 13. We also extend the scope of the theorem to a wider range of categories \mathcal{K} , noting that every accessible category may be embedded as a full subcategory of $\mathbf{Str} \Sigma$ (see for example [1, Characterization Theorem 5.35]).

We use the convention of [3], calling a category \mathcal{C} Σ_n definable if there is a Σ_n formula $\varphi(x_1, \dots, x_8)$ (in the language of set theory) and a parameter p such that $\varphi(A, B, C, f, g, h, i, p)$ holds if and only if A, B and C are objects of \mathcal{C} , $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$, $h \in \mathcal{C}(A, C)$, $h = g \circ f$ and $i = \text{Id}_A$. Note that from such a φ , formulas may be obtained for $\text{Obj}(\mathcal{C})$, $\text{Mor}(\mathcal{C})$, \circ and Id , with the only extra quantification an $\exists i$ for some of them. We say that a functor $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ is Σ_n definable if there are Σ_n formulas $\varphi_{\text{Obj}}^F(x, y)$ and $\varphi_{\text{Mor}}^F(x, y)$ such that for any object A and morphism f of \mathcal{C}_0 , $\varphi_{\text{Obj}}^F(A, B)$ holds if and only if $B = F(A)$, and $\varphi_{\text{Mor}}^F(f, g)$ holds if and only if $g = F(f)$.

An annoying quirk of using infinitary languages is that $\mathbf{Str} \Sigma$ need not be absolute: for example, a function defined on all countably infinite tuples from a set X in some set theoretic universe will no longer be defined on all countably infinite tuples of X if we move to a universe with more countably infinite subsets of X . However, this obstacle, generalised to arbitrary infinite cardinalities, is the only obstruction to absoluteness.

Proposition 16. *Let Σ be a signature. If Σ contains no infinitary function symbols, then $X \in \text{Obj}(\mathbf{Str} \Sigma)$ is Δ_1 definable with Σ as a parameter; otherwise, $X \in$*

$\text{Obj}(\mathbf{Str} \Sigma)$ is Π_1 definable with parameter Σ . Moreover, if for some κ greater than the arities of all the function symbols in Σ , we add the function \mathcal{P}_κ to the language of set theory (where $\mathcal{P}_\kappa(X)$ denotes the set of all subsets of X of cardinality less than κ), then $X \in \text{Obj}(\mathbf{Str} \Sigma)$ is Δ_0 definable for this extended language (again with Σ as a parameter).

The point is that Δ_1 definability implies absoluteness between models of set theory, whereas Π_1 definability only implies downward absoluteness. The second part of the Proposition tells us that we have absoluteness of $\text{Obj}(\mathbf{Str} \Sigma)$ between models of set theory that agree about the function \mathcal{P}_κ .

Proof. With the precise definition of the notion of a Σ -structure from Section 2, it is straightforward to show that $A \in \text{Obj}(\mathbf{Str} \Sigma)$ is Δ_0 in the parameters Σ and $\mathcal{P}_\kappa(A)$, where κ is greater than all of the arities of symbols in Σ . It is well known that $\mathcal{P}_{\aleph_0}(A)$ is Δ_1 definable from A , but for greater κ it is Π_1 definable with parameter κ ; since it is unique, adding its definition to the formula makes the expression Δ_1 in the first case (see the comment on page 9 where we defined Δ_1) and Π_1 in the second. Moreover, for each λ -ary relation symbol R , one just needs to verify that each element of R^A is a function from λ to A , and so rather than $\mathcal{P}_{\lambda^+}(A)$ as a parameter it suffices for the definition to just have λ , which is recoverable from R itself. \square

Note that the initial segments V_λ of V , which are relevant to $C^{(n)}$ -extendible and $C^{(n)+}$ -extendible cardinals, are *correct* for \mathcal{P}_κ : for any X in V_λ for λ a limit ordinal, V_λ contains every subset of X of cardinality less than κ (and indeed, every subset of X of any cardinality), and so V_λ agrees with V about the function \mathcal{P}_κ . Thus, a $C^{(n)}$ -extendible cardinal κ has embeddings witnessing its $C^{(n)}$ -extendibility that are actually elementary for formulas in the language of set theory extended by \mathcal{P}_κ (and likewise for $C^{(n)+}$ -extendibility).

We claimed above that Theorem 18 is more general than Theorem 13. Certainly in a ZFC setting every category (indeed every class) is definable, and so Σ_n -definable for some n . In Theorem 18 we require \mathcal{K} to be a full subcategory of $\mathbf{Str} \Sigma$ for some signature Σ , but it turns out that this still allows a vast array of categories, including all accessible ones: we have the following characterisation of accessible categories (see [1, Theorem 5.35]).

Theorem 17. *Accessible categories are precisely the categories equivalent to categories of models of basic theories, that is, those whose formulas are universally quantified implications of positive-existential formulas.*

Theorem 18. *Suppose that $n > 0$, \mathcal{K} is a full subcategory of $\mathbf{Str} \Sigma$ for some signature Σ , and $F : \mathcal{A} \rightarrow \mathcal{K}$ is a Σ_n definable full embedding with Σ_n definable domain category \mathcal{A} . If there exists a $C^{(n)}$ -extendible cardinal greater than the rank of Σ , the arity of each function or relation symbol in Σ , and the rank of the parameters involved in some Σ_n definitions of F and \mathcal{A} and some definition of \mathcal{K} , then there exists a regular cardinal λ such that F preserves λ -directed colimits.*

Proof. By Lemma 6 it suffices to show that the embedding $i \circ F : \mathcal{A} \rightarrow \mathbf{Str} \Sigma$ preserves λ -directed colimits for some regular cardinal λ , where i is the inclusion functor from \mathcal{K} to $\mathbf{Str} \Sigma$. Note that if F is Σ_n definable as a functor to \mathcal{K} then it remains so as a functor to $\mathbf{Str} \Sigma$, that is, $i \circ F$ is Σ_n definable. Thus, let us assume without loss of generality that $\mathcal{K} = \mathbf{Str} \Sigma$. In particular, we may use the fact that

$\mathbf{Str} \Sigma$ has all λ -directed colimits for λ greater than all of the arities of symbols in Σ .

For terminological convenience let β be a cardinal greater than the rank of Σ , the arity of each of the symbols in Σ , and the rank of the parameters involved in the definitions of F and \mathcal{A} . Let \mathcal{C} be the category whose objects are maps $a : \bar{A} \rightarrow F(A)$, where, for some regular cardinal $\lambda > \beta$ and some λ -directed diagram \mathcal{D} in \mathcal{A} , A is a colimit in \mathcal{A} of \mathcal{D} , \bar{A} is a colimit of $F\mathcal{D}$ in $\mathbf{Str} \Sigma$, and a is the homomorphism induced by the image of the \mathcal{A} -colimit cocone under F . The morphisms $f : a \rightarrow b$ are pairs $f = \langle g, h \rangle$, with $g \in \text{Hom}_{\mathbf{Str} \Sigma}(\bar{A}, \bar{B})$ and $h \in \text{Hom}_{\mathbf{Str} \Sigma}(F(A), F(B))$, such that the following diagram commutes:

$$\begin{array}{ccc} \bar{A} & \xrightarrow{a} & F(A) \\ g \downarrow & & \downarrow h \\ \bar{B} & \xrightarrow{b} & F(B). \end{array}$$

Let \mathcal{C}^* be the full subcategory of \mathcal{C} whose objects are those $a \in \text{Obj}(\mathcal{C})$ that are not isomorphisms. If the conclusion of the theorem fails, that is, if for every regular cardinal λ , some λ -directed colimit is not preserved by F , then even up to isomorphism \mathcal{C}^* has a proper class of objects: in the terminology of [1, Section 0.1], \mathcal{C}^* is not essentially small.

We claim that membership in $\text{Obj}(\mathcal{C}^*)$ is Σ_{n+2} definable over the language of set theory extended by \mathcal{P}_β . We have: $a \in \text{Obj}(\mathcal{C}^*)$ if and only if

$$\begin{aligned} \exists \lambda \exists \mathcal{D} \exists \langle \bar{A}, \bar{\eta} \rangle \exists \langle A, \eta \rangle & (\lambda \text{ is a regular cardinal} \wedge \mathcal{D} \text{ is a diagram in } \mathcal{A} \wedge \\ & \mathcal{D} \text{ is } \lambda\text{-directed} \wedge \\ & \langle \bar{A}, \bar{\eta} \rangle = \text{Colim}_{\mathbf{Str} \Sigma}(F\mathcal{D}) \wedge \langle A, \eta \rangle = \text{Colim}_{\mathcal{A}}(\mathcal{D}) \wedge \\ & a : \bar{A} \rightarrow F(A) \text{ is the induced homomorphism} \wedge \\ & a \text{ is not an isomorphism}). \end{aligned}$$

Here we are treating a diagram \mathcal{D} as a set of objects and morphisms, and we use $\langle A, \eta \rangle = \text{Colim}_{\mathcal{A}}(\mathcal{D})$ to mean that A is the colimit in \mathcal{A} of \mathcal{D} with colimit cocone η . The statement “ a is not an isomorphism” is Δ_0 in \bar{A} and A , “ λ is a regular cardinal” and “ \mathcal{D} is λ -directed” are Π_1 , and “ \mathcal{D} is a diagram in \mathcal{A} ” is Σ_n since \mathcal{A} is. The statement that “ $a : \bar{A} \rightarrow F(A)$ is the induced homomorphism” can simply be expressed by saying that a is a homomorphism from \bar{A} to $F(A)$ and the requisite triangles with cocone maps (indexed by objects of \mathcal{D}) commute, so this part of the formula is just Σ_n because F needs to be evaluated for it. In terms of quantifier complexity the crux is really the statement $\langle A, \eta \rangle = \text{Colim}_{\mathcal{A}}(\mathcal{D})$, as this is equivalent to saying that for every cocone over \mathcal{D} in \mathcal{A} there is a unique morphism in \mathcal{A} from A to the vertex object of that cocone making everything relevant commute; this can be expressed by a Π_{n+1} formula, handling existence and uniqueness separately to save on quantifiers. Similarly, $\langle \bar{A}, \bar{\eta} \rangle = \text{Colim}_{\mathbf{Str} \Sigma}(F\mathcal{D})$ is Π_{n+1} .

Assume for the sake of obtaining a contradiction that \mathcal{C}^* is not essentially small. Let κ be a $C^{(n)}$ -extendible cardinal greater than β so that embeddings with critical point κ do not affect the parameters in the definitions of $\mathbf{Str} \Sigma$, \mathcal{A} or F . By Proposition 14, we may assume without loss of generality that κ is in fact $C^{(n)+}$ -extendible. Let a be an object of \mathcal{C}^* of rank greater than κ , arising from a λ_a -directed diagram \mathcal{D}_a for some λ_a also greater than κ . Let $\lambda \in C^{(n)}$ be greater

than the ranks of a , \mathcal{D}_a , $F\mathcal{D}_a$, and the corresponding colimit cocones $\langle \bar{A}, \bar{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$; in particular, sufficiently large that V_λ contains witnesses to the fact that $a \in \text{Obj}(\mathcal{C}^*)$. Let $j : V_\lambda \rightarrow V_\mu$ be an elementary embedding with critical point κ , such that $\mu > j(\kappa) > \lambda$ are all in $C^{(n)}$. In particular,

$$V_\mu \models \text{"}\lambda_a, \mathcal{D}_a, \langle \bar{A}, \bar{\eta} \rangle_a, \text{ and } \langle A, \eta \rangle_a \text{ witness that } a \in \text{Obj}(\mathcal{C}^*)\text{"},$$

since the statement in quotes is Π_{n+1} , and hence downwardly absolute from V to Σ_n -correct sets such as V_μ which contain all of the parameters. Of course, not everything that V_μ believes to be in $\text{Obj}(\mathcal{C}^*)$ need be so in V . However, it will suffice for our purposes to carry out the remainder of the argument within V_μ , obtaining a contradiction from the fact that a is not an isomorphism (whether construed in V_μ or V). We use the standard notation of using a superscript V_μ to indicate that an expression is to be interpreted in V_μ ; thus for example \mathcal{C}^{*V_μ} denotes the set $\{x \in V_\mu \mid V_\mu \models x \in \mathcal{C}^*\}$. Also note that, as they have Σ_n definitions, membership in \mathcal{A} and the evaluation of F are in any case absolute to V_μ .

By the choice of $\kappa > \beta$, we have that j is the identity on the parameters to the definition of F , and so j commutes with F . Since j is elementary, we have a morphism of \mathcal{C}^{*V_μ}

$$\begin{array}{ccc} \bar{A} & \xrightarrow{a} & F(A) \\ j \upharpoonright \bar{A} \downarrow & & \downarrow j \upharpoonright F(A) \\ j(\bar{A}) & \xrightarrow{j(a)} & j(F(A)), \end{array}$$

which we denote by $j \upharpoonright a : a \rightarrow j(a)$. Indeed, by elementarity $j(a)$ is the induced homomorphism from $j(\bar{A})$ (the colimit in $\mathbf{Str} \Sigma$ of $j(F\mathcal{D}_a)$) to $j(F(A)) = F(j(A))$. Further, $j(a)$ is not an isomorphism, and so lies in \mathcal{C}^{*V_μ} . The commutativity of the diagram also follows by elementarity, as in Lemma 5: for any element α of the Σ -structure \bar{A} , $j(a) \circ j \upharpoonright \bar{A}(\alpha) = j(a)(j(\alpha)) = j(a(\alpha)) = (j \upharpoonright F(A) \circ a)(\alpha)$.

Since $j(\lambda_a) > j(\kappa) > \lambda$ and $j(\mathcal{D}_a)$ is $j(\lambda_a)$ -directed, $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is certainly λ -directed. The set of objects $j \text{ "Obj}(F\mathcal{D}_a)$ is a subset of cardinality less than λ of the objects of $Fj(\mathcal{D}_a)$, and hence has an upper bound $F(d_0)$ in $Fj(\mathcal{D}_a)$. Composing the cocone from $j \text{ " } F\mathcal{D}_a$ to $F(d_0)$ with the natural transformation $j \upharpoonright \cdot$ from $F\mathcal{D}_a$ to $j \text{ " } (F\mathcal{D}_a)$ (see Lemma 5), we get a cocone from $F\mathcal{D}_a$ to $F(d_0)$. The picture is essentially the same as shown in diagram (*) in the proof of Theorem 10.

$$\begin{array}{ccccc} & \bar{A} & & j \upharpoonright \bar{A} & \\ & \nearrow & & \searrow & \\ & \bar{A} & \xrightarrow{g_{\bar{A}}} & F(d_0) & \xrightarrow{j(\delta^{\bar{A}})_{F(d_0)}} & j(\bar{A}) \\ & \downarrow a & \nearrow F(g_A) & \downarrow j \upharpoonright F(A) & \downarrow j(a) & \nearrow j(\delta^{\bar{A}}) \\ & F(A) & \xrightarrow{j \upharpoonright F(A)} & F(j(A)) & & \\ & \nearrow F\delta^A & \nearrow (j \upharpoonright \cdot) & \nearrow \text{inclusion} & \nearrow Fj(\delta^A) & \\ F\mathcal{D}_a & \xrightarrow{\quad} & F(j \text{ " } \mathcal{D}_a) & \xrightarrow{\quad} & F(j(\mathcal{D}_a)) \end{array}$$

As in the proof of Theorem 10, we let $g_{\bar{A}} : \bar{A} \rightarrow F(d_0)$ and $F(g_A) : FA \rightarrow F(d_0)$ be the maps induced by the cocone from $F\mathcal{D}$ to $F(d_0)$. Since the maps $j \upharpoonright \bar{A} : \bar{A} \rightarrow$

$j(\bar{A})$ and $j \upharpoonright F(A) : F(A) \rightarrow F(j(A))$ make the diagrams with the corresponding cocones commute, they must be the induced colimit maps.

We may now deduce that a is an isomorphism. By elementarity once again, the homomorphism $j \upharpoonright \bar{A} : \bar{A} \rightarrow j(\bar{A})$ is injective and preserves the complements of the relations in Σ ; thus, since it factors through a as $j \upharpoonright \bar{A} = j(\delta^{\bar{A}}_{F(d_0)}) \circ F(g_A) \circ a$, the same is true of a . It therefore only remains to show that a is surjective. If we had $\alpha \in F(A) \setminus a''\bar{A}$, then by elementarity $j(\alpha)$ would lie in $F(j(A)) \setminus j(a)''j(\bar{A})$, contradicting the fact that $j(\alpha) = j(a) \circ \delta^{\bar{A}}_{F(d_0)} \circ F(g_A)(\alpha)$. Hence, a is indeed an isomorphism in V_μ . But this contradicts the definition of \mathcal{C}^{*V_μ} , and so returning to our initial assumption, we may conclude that (in V) \mathcal{C}^* is essentially small, as required. \square

In light of Theorem 18, it is natural to ask whether Theorem 10 can be generalised. Central to the proof of Theorem 10 was the absoluteness of $\mathbf{Mod} T$ between different models of set theory (again, with the \mathcal{P}_κ function added to the language of set theory for κ greater than the arities of the symbols in Σ). This absoluteness arises from the fact that $\mathbf{Mod} T$ is Δ_1 definable in the parameters Σ and T , and it turns out that for such categories a generalisation is indeed possible, using the same large cardinal assumption far below $C^{(n)}$ -extendible cardinals in strength.

Theorem 19. *Suppose that \mathcal{K} is a definable full subcategory of $\mathbf{Str} \Sigma$ for some signature Σ , and \mathcal{A} is a Δ_1 definable full subcategory of \mathcal{K} . If there exists an α -strongly compact cardinal κ for some α greater than the ranks of the parameters in a Δ_1 definition of \mathcal{A} and a definition of \mathcal{K} , then the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{K}$ preserves κ -directed colimits.*

Proof. The proof exactly follows that of Theorem 10 to build up the analogue of diagram (*), using the absoluteness of Δ_1 definitions between models of set theory. From that point, whilst we cannot use an elementary chains argument in this context, the argument from the proof of Theorem 18 that the morphism from the \mathcal{K} -colimit to the \mathcal{A} -colimit is an isomorphism *does* translate. \square

As already mentioned above, the conclusion of Theorem 13 is in fact equivalent to VP as shown by Example 6.12 of [1]. The example may also be stratified, to show that some degree of large cardinal strength is necessary for the conclusion of Theorem 18.

Theorem 20. *For any $n \geq 1$, suppose that for every signature Σ , every Π_{n+1} definable full subcategory \mathcal{K} of $\mathbf{Str} \Sigma$, and every Π_{n+1} definable full embedding $F : \mathcal{A} \rightarrow \mathcal{K}$, there is a regular cardinal λ such that F preserves λ -directed colimits. Then there exists a proper class of $C^{(n)}$ -extendible cardinals.*

Proof. We prove the contrapositive by means of a counterexample. In [2] it is shown that the existence of a proper class of $C^{(n)}$ -extendible cardinals is equivalent to Vopěnka's Principle for Σ_{n+2} classes of structures. Indeed, the proof of Theorem 4.12 of [2] exhibits, under the assumption that there are no $C^{(n)}$ -extendible cardinals, a Π_{n+1} class of structures, between distinct elements of which there can be no elementary embeddings. If there is some bound β such that all $C^{(n)}$ -extendible cardinals are less than β , then the same construction can be employed starting from β to again give a Π_{n+1} class with no elementary embeddings between distinct elements. Changing the construction slightly to give each structure an underlying

set of the form $V_{\lambda_\alpha+2}$ further eliminates the possibility of non-trivial elementary embeddings from one of the structures to itself by Kunen's inconsistency theorem [8], without changing the complexity of the definition. Another useful feature of the construction is that the structures are over a finitary relational signature Σ .

So suppose we have such a Π_{n+1} class \mathcal{C} of Σ -structures with no elementary embeddings for some finitary relational Σ . We expand Σ to a signature Σ' by adding *Skolem relations*. Using Gödel numbering, we may take a recursive bijection $i \mapsto \varphi_i$ between natural numbers and Σ -formulas. Moreover, this may be done in such a way that φ_i has at most i free variables, and by ignoring extra values we may treat φ_i as having exactly i free variables. We take $\Sigma' = \Sigma \cup \{R_i \mid i \in \omega\}$, where for each i , R_i is an i -ary relation symbol. For each structure $M \in \mathcal{C}$ we take an expanded structure M' given by M concatenated with $\langle R_i^M \mid i \in \omega \rangle$, where for each i and M , we set

$$R_i^M = \{(m_1, \dots, m_i) \in M^i \mid M \models \varphi_i(m_1, \dots, m_i)\}.$$

The satisfaction relation \models is known to be Δ_1 for finitary languages, and so the class $\mathcal{C}' = \{M' \mid M \in \mathcal{C}\}$ remains Π_{n+1} . Moreover, \mathcal{C}' admits no non-trivial (Σ') -homomorphisms between its elements, as a homomorphism $h : M' \rightarrow N'$ must restrict to an elementary embedding $M \rightarrow N$.

With such a class \mathcal{C}' to hand, the argument now proceeds very much as for [1, Example 6.12]. Let \mathcal{A} be the full subcategory of Σ' -structures consisting of those Σ' -structures A such that $\text{Hom}(C, A)$ is empty for all $C \in \mathcal{C}'$, as well as the terminal object T (a single point Σ' -structure, with $R^T = T^i$ for each i -ary relation R). The only unbounded quantifiers this definition uses to build up from that of \mathcal{C}' are universal, so \mathcal{A} is also Π_{n+1} , and the inclusion of \mathcal{A} into $\mathbf{Str} \Sigma'$ is therefore also Π_{n+1} . For each regular cardinal λ , consider $C \in \mathcal{C}'$ of cardinality at least λ . Clearly each proper substructure of C lies in \mathcal{A} , so we may consider the λ -directed diagram \mathcal{D} of all substructures of C of cardinality strictly less than λ (since Σ' is relational, these are just the $< \lambda$ -sized subsets with the induced structure). In $\mathbf{Str} \Sigma'$, the colimit of \mathcal{D} is C , so by the definition of \mathcal{A} , the only cocone on \mathcal{D} in \mathcal{A} is the cocone to the terminal object. Thus, $\text{Colim}_{\mathcal{A}}(\mathcal{D}) = T \neq C$, so the inclusion functor does not preserve λ -directed colimits. \square

The complexities in Theorems 18 and 20 are such that we lose strength moving from large cardinals to colimit preservation and back again. It would be very interesting to improve one or both of these Theorems to close this gap.

One might also hope to draw large cardinal strength at the bottom of the definability hierarchy from the equivalence shown in [3] between Vopěnka's Principle for Δ_2 classes and the existence of a proper class of supercompact cardinals. However, a naïve modification of the proof of Theorem 20 is fruitless, as the extra universal quantifier involved in the definition of \mathcal{A} would take us up to Π_2 , for which the $n = 1$ case of Theorem 20 is already a better result.

However, we now show that one *can* obtain large cardinal strength from a Δ_2 example, which moreover is a less contrived example than those used to prove Theorem 20. For this we use the notion of a *group radical*.

Definition 21. *For any abelian group X , the radical singly generated by X is the functor R_X from abelian groups to abelian groups given by*

$$R_X(G) = \bigcap_{f \in \text{Hom}(G, X)} \ker(f),$$

with the action on homomorphisms simply given by restricting. For any cardinal κ , we define the functor R_X^κ by

$$R_X^\kappa(G) = \sum_{A \leq G, |A| < \kappa} R_X(A),$$

where as usual $A \leq G$ denotes that A is a subgroup of G , and again we take restrictions for homomorphisms.

Note that if A is a subgroup of G then $R_X(A) \subseteq A \cap R_X(G)$, and so $R_X^\kappa(G)$ is a subgroup of $R_X(G)$. In [6] (see also [4]) it is shown how to draw large cardinal strength from group radical considerations.

Theorem 22 (see [4, Theorem 4.11.1]). *A cardinal κ is ω_1 -strongly compact if and only if $R_{\mathbb{Z}} = R_{\mathbb{Z}}^\kappa$.*

With this in place, our example is now remarkably straightforward. Note that group homomorphisms take radicals to radicals, so we can consider the category of abelian groups with their radical as a distinguished predicate, with group homomorphisms as the morphisms.

Theorem 23. *Let $\Sigma = \{\cdot, R\}$ with \cdot a binary operation symbol and R a unary predicate symbol. Let \mathcal{A} be the full subcategory of $\mathbf{Str} \Sigma$ whose objects are abelian groups G with the radical $R_{\mathbb{Z}}(G)$ as the interpretation of the predicate R . If the inclusion of \mathcal{A} in $\mathbf{Str} \Sigma$ preserves κ -directed colimits for some regular cardinal κ , then κ is ω_1 -strongly compact.*

Proof. We shall show that under the hypotheses of the theorem, $R_{\mathbb{Z}}(G) = R_{\mathbb{Z}}^\kappa(G)$ for every abelian group G . So suppose that $\mathcal{A} \hookrightarrow \mathbf{Str} \Sigma$ preserves κ -directed colimits for some regular cardinal κ , and let G be an arbitrary abelian group. Let \mathcal{D} be the diagram of all of the Σ -structures $(A, R_{\mathbb{Z}}(A))$ with A a subgroup of G of cardinality less than κ , and inclusions as the morphisms of the diagram. The colimit of this diagram in $\mathbf{Str} \Sigma$ is as ever the direct limit, which is easily seen to be $(G, R_{\mathbb{Z}}^\kappa(G))$. However, the colimit in \mathcal{A} must be $(G, R_{\mathbb{Z}}(G))$. Indeed, if we temporarily forget the predicate for the radical, G is clearly the colimit, and then since radicals are respected by all group homomorphisms, we see that $(G, R_{\mathbb{Z}}(G))$ satisfies the requirements to be the colimit in \mathcal{A} . Thus, colimit preservation tells us that in fact $R_{\mathbb{Z}}(G) = R_{\mathbb{Z}}^\kappa(G)$, and so by Theorem 22, κ is ω_1 -strongly compact. \square

Since $h \in R_{\mathbb{Z}}(G)$ is Π_1 and consequently $h \notin R_{\mathbb{Z}}(G)$ is Σ_1 , we have that $h \in \mathcal{I}(R) \iff h \in R_{\mathbb{Z}}(G)$ is Δ_2 . Hence, as alluded to above, the category \mathcal{A} of Theorem 23 is Δ_2 definable.

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